JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 99, 338–349 (1984)

Open Maps, Closed Maps and Local Compactness in Fuzzy Topological Spaces

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1. INTRODUCTION

Chang [2] defined fuzzy topological spaces by using fuzzy sets introduced by Zadeh [14]. Since then a number of research papers have been published. Fuzzy continuous functions were defined by Chang. Not much is known about the fuzzy open and fuzzy closed maps. A few characterizations of such maps were given in [8].

In this paper some additional characterizations of these maps are given and it is shown that the normality as defined by Hutton [7] is invariant under F-closed, F-continuous maps. By using α-compactness of Gantner *et al.* [5], we have defined α-perfect maps in fuzzy topological spaces and shown that many results of perfect maps in general topology [4] are extendable to fuzzy topology. For example, the 1*-Hausdorff and regularity axioms are invariant under 1*-perfect maps.

The localization of compactness in fuzzy topological spaces was initiated by Wong [13] and continued by Christoph [3] and Gantner *et al.* [5]. Since the theory of α-compactness due to Gantner *et al.* [5] is the most satsifactory theory of compactness in fuzzy topology, the local α-compactness due to them is more natural than the definitions given by Wong and Christoph. We have defined local α-compactness using a weaker hypothesis than that given in [5]. We have shown that the local α-compactness is productive and is invariant under F-continuous, F-open maps.

2. PRELIMINARIES

The definitions of fuzzy sets, fuzzy topological spaces (abb. FTS) and neighborhoods, etc., are found in [2, 10, 14]; the definitions of shading

* The Research was supported by a Fellowship of the University Grants Commission, New Delhi, India.

0022-247X/84 $3.00

families, α-compactness, and α^*-compactness are found in [5], the definitions of α-Hausdorff and α^*-Hausdorff FTS are found in [9].

We would like to mention the following basic theorems from [2, 10].

THEOREM 2.1 [2]. *Let f be a function from a set X into a set Y.*

(1) *If $a \leqslant b$, then $f(a) \leqslant f(b)$ for any two fuzzy sets a, b, in X.*
(2) *If $c \leqslant d$, then $f^{-1}(c) \leqslant f^{-1}(d)$ for any two fuzzy sets c, d, in Y.*
(3) *$a \leqslant f^{-1}[f(a)]$ for any fuzzy set a in X.*
(4) *$b \geqslant f[f^{-1}(b)]$ for any fuzzy set b in Y.*
(5) *$f^{-1}(1-b) = 1 - f^{-1}(b)$ for any fuzzy set b in Y.*
(6) *$f(1-a) \geqslant 1 - f(a)$ for any fuzzy set a in X.*

THEOREM 2.2 [10]. *Let f be a function from a set X into a set Y. If a and a_i, $i \in I$ are fuzzy sets in X and b and b_j, $j \in J$ are fuzzy sets in Y, then the following relations hold.*

(1) *$f[f^{-1}(b)] = b$ when f is onto.*
(2) $f(\bigwedge a_i) \leqslant \bigwedge f(a_i)$.
(3) $f^{-1}(\bigwedge b_j) = \bigwedge f^{-1}(b_j)$.
(4) $f(\bigvee a_i) = \bigvee f(a_i)$.
(5) $f^{-1}(\bigvee b_j) = \bigvee f^{-1}(b_j)$.
(6) $f[f^{-1}(b) \wedge a] = b \wedge f(a)$.

DEFINITION 2.3 [2]. Let f be a function from a FTS (X, T) to a FTS (Y, S). Then

(1) f is said to be F-continuous iff for each $b \in S$, $f^{-1}(b) \in T$ or equivalently for each closed fuzzy set h in (Y, S), $f^{-1}(h)$ is a closed fuzzy set in (X, T).

(2) f is F-open (F-closed) iff for each open (closed) fuzzy set a in (X, T), $f(a)$ is open (closed) fuzzy set in (Y, S).

Warren [11] defined the fuzzy topological boundary of a fuzzy set in a FTS (X, T), denoted by $\mathrm{Fr}(a)$ and proved that $\mathrm{Fr}(a) \geqslant \bar{a} - a^0$ and inequality is strict in some cases.

3. OPEN MAPS AND CLOSED MAPS

In [8] some characterizations of open and closed maps in fuzzy topological spaces are already obtained. In the next two theorems we give some more equivalent characterizations of these maps.

THEOREM 3.1. *Let f be a function from a* FTS (X, T) *to a* FTS (Y, S). *Then the statements* (1)–(4) *are equivalent and* (4) *implies* (5).

(1) *f is F-open.*

(2) $f(a^0) \leqslant [f(a)]^0$ *for each fuzzy set a in X.*

(3) $f^{-1}(\bar{b}) \leqslant \overline{f^{-1}(b)}$ *for each fuzzy set b in Y.*

(4) $[f^{-1}(b)]^0 \leqslant f^{-1}(b^0)$ *for each fuzzy set b in Y.*

(5) $f^{-1}(\bar{b} - b^0) \leqslant Fr[f^{-1}(b)]$ *for any fuzzy set b in Y.*

Proof. (1) $\Leftrightarrow$ (2): *Proved in* [8].

(2) $\Rightarrow$ (3): Let b be a fuzzy set in Y. Let $a = f^{-1}(1-b)$. From (2), $f(a^0) \leqslant [f(a)]^0 \leqslant (1-b)^0$. Hence from [10, Theorem 2.13] and Theorem 2.1, $[f^{-1}(1-b)]^0 = a^0 \leqslant f^{-1}[f(a^0)] \leqslant f^{-1}[(1-b)^0]$. Then $f^{-1}(\bar{b}) = f^{-1}[1-(1-\bar{b})] = f^{-1}[1-(1-b)^0] = 1 - f^{-1}[(1-b)^0] \leqslant 1 - [f^{-1}(1-b)]^0 = \overline{1 - f^{-1}(1-b)} = \overline{f^{-1}[1-(1-b)]} = \overline{f^{-1}(b)}$.

(3) $\Rightarrow$ (4): Let b be any fuzzy set in Y and $c = 1 - b$. From (3), $f^{-1}(\bar{c}) \leqslant \overline{f^{-1}(c)}$. By [10, Theorem 2.13] and Theorem 2.1, $[f^{-1}(b)]^0 = [f^{-1}(1-c)]^0 = [1 - f^{-1}(c)]^0 = 1 - \overline{f^{-1}(c)} \leqslant 1 - f^{-1}(\bar{c}) = f^{-1}(1-\bar{c}) = f^{-1}[(1-c)^0] = f^{-1}(b^0)$.

(4) $\Rightarrow$ (2): Let a be a fuzzy set in X and $b = f(a)$. Then from (4) and by [10, Theorem 2.13] and Theorem 2.1, we have $a^0 \leqslant [f^{-1}(f(a))]^0 = [f^{-1}(b)]^0 \leqslant f^{-1}(b^0)$. Hence, $f(a^0) \leqslant f[f^{-1}(b^0)] \leqslant b^0 = [f(a)]^0$.

(4) $\Rightarrow$ (5): Let b be any fuzzy set in Y. From (4) $[f^{-1}(b)]^0 \leqslant f^{-1}(b^0)$. Since (4) $\Leftrightarrow$ (3), we also have $f^{-1}(\bar{b}) \leqslant \overline{f^{-1}(b)}$. Hence $f^{-1}(\bar{b}) - f^{-1}(b^0) \leqslant \overline{f^{-1}(b)} - [f^{-1}(b)]^0$ or $f^{-1}(\bar{b} - b^0) \leqslant \mathrm{Fr}(f^{-1}(b))$. This completes the proof.

THEOREM 3.2. *Let f be a map from a* FTS X *into a* FTS Y. *Then f is F-closed (F-open) iff for each fuzzy set a in Y and for any open (closed) fuzzy set b in X such that $f^{-1}(a) \leqslant b$, there is an open (closed) fuzzy set c in Y such that $a \leqslant c$ and $f^{-1}(c) \leqslant b$.*

Proof. Suppose f is F-closed. Let a be a fuzzy set in Y and let b be an open fuzzy set in X such that $f^{-1}(a) \leqslant b$. Let $c = 1 - f(1-b)$. Then c is an open fuzzy set in Y. Hence, $1 - b \leqslant 1 - f^{-1}(a) = f^{-1}(1-a)$. By Theorem 2.1, $f(1-b) \leqslant f[f^{-1}(1-a)] \leqslant 1 - a$, which implies $a \leqslant 1 - f(1-b) = c$. Further $f^{-1}(c) = f^{-1}[1 - f(1-b)] = 1 - f^{-1}[f(1-b)] \leqslant 1 - (1-b) = b$, again by Theorem 2.1.

Next suppose that f satisfies the condition of the theorem. We show that f is F-closed. Let b be a closed fuzzy set in X. Then $a = 1 - b$ is an open fuzzy set in X. Then $f^{-1}[1 - f(b)] = 1 - f^{-1}(f(b)) \leqslant 1 - b = a$. By hypothesis there is an open fuzzy set c in Y such that $1 - f(b) \leqslant c$ and $f^{-1}(c) \leqslant a = 1 - b$. Hence, $1 - c \leqslant f(b)$. Also $b \leqslant 1 - f^{-1}(c) = f^{-1}(1-c)$. By Theorem

2.1, we get $f(b) \leqslant f[f^{-1}(1-c)] \leqslant 1-c$. Thus we have $f(b) = 1-c$, which is a closed fuzzy set in Y and hence f is an F-closed map. The proof for the "open" case is obtained by interchanging the words "open" and "closed" in the above arguments.

Hutton [7] defined that a FTS X is normal if for every closed fuzzy set k and every open fuzzy set b such that $k \leqslant b$, there is a fuzzy set a such that $k \leqslant a^0 \leqslant \bar{a} \leqslant b$.

It is well known in general topology that normality is invariant under a closed, continuous surjection. This result is extended to fuzzy topological spaces in the following:

THEOREM 3.3. *Let f be an F-closed, F-continuous function from a normal* FTS *X onto a* FTS *Y. Then Y is normal.*

Proof. Let k be a closed fuzzy set in Y and b be an open fuzzy set in Y such that $k \leqslant b$. Then $f^{-1}(k)$ and $f^{-1}(b)$ are respectively closed and open fuzzy sets in X such that $f^{-1}(k) \leqslant f^{-1}(b)$. Since X is normal, there exists a fuzzy set a in X such that $f^{-1}(k) \leqslant a^0 \leqslant \bar{a} \leqslant f^{-1}(b)$. Since f is F-closed, by Theorem 3.2, there is an open fuzzy set g in Y such that $k \leqslant g$ and $f^{-1}(g) \leqslant a^0$. Now, since f is onto, $g = f(f^{-1}(g)) \leqslant f(a^0) \leqslant f(\bar{a}) \leqslant f[f^{-1}(b)] = b$ by Theorems 2.1 and 2.2. Since f is F-closed, $f(\bar{a})$ is a closed fuzzy set and $g \leqslant f(\bar{a})$ implies $\bar{g} \leqslant f(\bar{a})$. Hence $k \leqslant g^0 \leqslant \bar{g} \leqslant b$, since $g = g^0$. Thus Y is normal.

In [8] we have defined regularity in FTS and obtained some equivalent characterizations.

DEFINITION 3.4 [8]. A FTS (X, T) is said to be regular if for each $x \in X$ and a closed fuzzy set a with $a(x) = 0$, there exist open fuzzy sets g and h such that $g(x) = 1$, $a \leqslant h$ and $g \leqslant 1-h$.

It is further proved that a FTS (X, T) is regular iff for each $x \in X$ and $g \in T$ with $g(x) = 1$ there exists $h \in T$ with $h(x) = 1$ such that $h \leqslant \bar{h} \leqslant g$.

It is shown in [1] that regularity is invariant under an open, closed, continuous surjection. This also holds for fuzzy topological spaces.

THEOREM 3.5. *Let $f: X \to Y$ be an F-continuous, F-open, F-closed surjection. If X is regular* FTS, *then so is Y.*

Proof. Let $q \in Y$ and $p \in X$ such that $f(p) = q$. Let g be an open fuzzy set in Y such that $g(q) = 1$. Then $f^{-1}(g)(p) = g(f(p)) = 1$ and $f^{-1}(g)$ is an open fuzzy set in X. Since X is regular there is an open fuzzy set h in X such that $h(p) = 1$ and $\bar{h} \leqslant f^{-1}(g)$. Since f is F-open, $f(h)$ is an open fuzzy set such that $(f(h))(q) = 1$ and $f(h) \leqslant f(\bar{h}) \leqslant g$. Since f is also F-closed, $\overline{f(h)} \leqslant f(\bar{h})$. Hence $f(h)$ is an open fuzzy set in Y such that $f(h)(q) = 1$ and $\overline{f(h)} \leqslant g$. Thus Y is a regular FTS.

4. Perfect Maps

Perfect mappings in fuzzy topological spaces were first introduced by Christoph [3]. Accordingly, an F-closed F-continuous surjection $f: X \to Y$ is F-perfect if $f^{-1}(q)$ is compact (in the sense of Chang) for each fuzzy point [13, Definition 3.1] q in Y. We give the following natural generalization of perfect mappings which excludes the notion of "fuzzy point" and includes the more general notion of compactness, viz., α-compactness defined in [5].

Definition 4.1. Let $0 \leqslant \alpha < 1$ (resp. $0 < \alpha \leqslant 1$). An F-closed F-continuous function f from a FTS X onto a FTS Y is said to be α-perfect (resp. α^*-perfect) iff $f^{-1}(y)$ is α-compact (resp. α^*-compact) for each $y \in Y$.

The following Hausdorff separation axiom is due to Rodabaugh.

Definition 4.2 [9]. Let $0 \leqslant \alpha < 1$ (resp. $0 < \alpha \leqslant 1$). A FTS (X, T) is said to be α-Hausdorff (resp. α^*-Hausdorff) if for each x, y in X with $x \neq y$ there exist g, h in T such that $g(x) > \alpha (g(x) \geqslant \alpha)$, $h(y) > \alpha (h(y) \geqslant \alpha)$ and $g \wedge h = 0$.

We have the following.

Theorem 4.3. *Let f be a 1^*-perfect map from a* FTS (X, T) *onto a* FTS (Y, S). *Then*

(1) *If (X, T) is 1^*-Hausdorff, so is (Y, S).*

(2) *If (X, T) is regular* FTS, *so is (Y, S).*

Proof. (1) Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint 1^*-compact sets in X. Then from [9, Theorem 5.2(1)], there are g, h in T such that $g = 1$ on $f^{-1}(y_1)$, $h = 1$ on $f^{-1}(y_2)$ and $g \wedge h = 0$. Clearly $g \geqslant f^{-1}(y_1)$ and $h \geqslant f^{-1}(y_2)$. Since f is F-closed, by Theorem 3.2, there exist open fuzzy sets g_1 and h_1 in Y such that $g_1(y_1) = 1$, $h_1(y_2) = 1$ and $f^{-1}(g_1) \leqslant g$, $f^{-1}(h_1) \leqslant h$. Then $g_1 \wedge h_1 = 0$ and hence Y is 1^*-Hausdorff.

(2) Let $y \in Y$ and let g be an open fuzzy set in Y such that $g(y) = 1$. Then we have $f^{-1}(y) \leqslant f^{-1}(g)$. Now for each $x \in f^{-1}(y)$, $f^{-1}(g)(x) = g(f(x)) = g(y) = 1$. Since X is regular, there exists an open fuzzy set a in X such that $a(x) = 1$ and $\bar{a} \leqslant f^{-1}(g)$. Since $f^{-1}(y)$ is 1^*-compact, there exist a_1, $a_2, \ldots, a_k$ open fuzzy sets in X such that $f^{-1}(y) \leqslant \bigvee_{i=1}^{k} a_i$ and $\bar{a}_i \leqslant f^{-1}(g)$ for $i = 1, 2, \ldots, k$. Let $b = \bigvee_{i=1}^{k} a_i$, which is an open fuzzy set in X, and $f^{-1}(y) \leqslant b \leqslant \bar{b} \leqslant f^{-1}(g)$. Since f is F-closed, from Theorem 3.2, there exists an open fuzzy set h in Y such that $h(y) = 1$ and $f^{-1}(h) \leqslant b$. Then $h = f[f^{-1}(h)] \leqslant f(\bar{b}) \leqslant f[f^{-1}(g)] = g$. Since $f(\bar{b})$ is a closed fuzzy set greater than or equal to h, $h \leqslant \bar{h} \leqslant f(\bar{b}) \leqslant g$, and thus Y is regular.

We recall the following definitions from [8].

DEFINITION 4.4 [8]. Let (X, T) be a FTS and $0 \leqslant \alpha < 1$ $(0 < \alpha \leqslant 1)$. Let $\mathscr{U}$ and $\mathscr{V}$ be any two α-shadings (resp. α^*-shadings) of X. Then $\mathscr{U}$ is said to be a refinement of $\mathscr{V}$ if for each $g \in \mathscr{U}$ there exists $h \in \mathscr{V}$ such that $g \leqslant h$.

DEFINITION 4.5 [8]. A family $\{a_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in a FTS (X, T) is said to be locally finite if for each x in X there exists an open fuzzy set g with $g(x) = 1$ such that $a_\lambda \leqslant 1 - g$ holds for all but at most finitely many $\lambda \in \Lambda$.

DEFINITION 4.6 [8]. Let $0 \leqslant \alpha < 1$ (resp. $0 < \alpha \leqslant 1$). A FTS (X, T) is said to be α-paracompact (resp. α^*-paracompact) if each α-shading (resp. α^*-shading) of X by open fuzzy sets has a locally finite α-shading (resp. α^*-shading) refinement by open fuzzy sets.

The definitions of countable α-compactness and α-Lindelöf property are obtained by modifying the definition of α-compactness in the obvious way [8].

We have the following:

THEOREM 4.7. *Let $f: X \to Y$ be a 1^*-perfect map.*

(1) *If Y is 1^*-compact, then X is 1^*-compact.*

(2) *If Y is 1^*-Lindelöf, then X is 1^*-Lindelöf.*

(3) *If Y is countably 1^*-compact, then X is countably 1^*-compact.*

(4) *If Y is 1^*-paracompact, then X is 1^*-paracompact.*

Proof. (1) Let $\{g_\lambda : \lambda \in \Lambda\}$ be any open 1^*-shading of X. Let $y \in Y$. Then $f^{-1}(y)$ is 1^*-compact and $\{g_\lambda : \lambda \in \Lambda\}$ is 1^*-shading of $f^{-1}(y)$ and therefore has a finite 1^*-subshading, say $\{g_\lambda : \lambda \in M(y)\}$, where $M(y)$ is a finite subset of Λ. Now since $f^{-1}(y) \leqslant \bigvee_{\lambda \in M(y)} g_\lambda$, and f is F-closed, by Theorem 3.2 there is an open fuzzy set h_y in Y such that $h_y(y) = 1$ and $f^{-1}(h_y) \leqslant \bigvee_{\lambda \in M(y)} g_\lambda$. Now $\{h_y : y \in Y\}$ is an open 1^*-shading of Y and since Y is 1^*-compact, it has a finite 1^*-subshading, say $\{h_y : y \in A\}$, where A is a finite subset of Y. Let $M = \bigcup_{y \in A} M(y)$. Then M is a finite subset of Λ and further, $\{g_\lambda : \lambda \in M\} = \{g_\lambda : \lambda \in M(y), y \in A\}$ is a 1^*-subshading of X and thus X is 1^*-compact.

(2) Let $\{g_\lambda : \lambda \in \Lambda\}$ be an open 1^*-shading of X. One can construct, as in (1), an open 1^*-shading $\{h_y : y \in Y\}$ of Y. Since Y is 1^*-Lindelöf, $\{h_y : y \in Y\}$ has a countable 1^*-subshading, say $\{h_y : y \in A\}$, where A is a countable subset of Y. Then the family $\{g_\lambda : \lambda \in M(y), y \in A\}$ is a countable 1^*-subshading of X and thus X is 1^*-Lindelöf.

(3) Let $\{g_n : n \in N\}$ be a countable open 1^*-shading of X. Let $h_n = 1 - f(1 - \bigvee_{i=1}^n g_i)$. Now $\{h_n : n \in N\}$ is a 1^*-shading of Y. For if $y \in Y$

then $f^{-1}(y)$ is 1*-compact and hence there is a finite 1*-subshading of $\{g_n : n \in N\}$, say $\{g_i : i = 1, 2, \ldots, m\}$ of $f^{-1}(y)$. Then we have $f^{-1}(y) \leqslant \bigvee_{i=1}^{m} g_i$ and then $(1 - \bigvee_{i=1}^{m} g_i) \leqslant 1 - f^{-1}(y) = f^{-1}(1 - \mu_y)$, where μ_y denotes the characteristic function of $\{y\}$. This implies that $f(1 - \bigvee_{i=1}^{m} g_i) \leqslant f(f^{-1}(1 - \mu_y)) \leqslant 1 - \mu_y$ and then $\mu_y \leqslant 1 - f(1 - \bigvee_{i=1}^{m} g_i) = h_m$ and hence $h_m(y) = 1$ and thus $\{h_n : n \in N\}$ is a 1*-shading of Y. Now since Y is countably 1*-compact, $\{h_n : n \in N\}$ has a finite 1*-subshading, say $\{h_i : i = 1, 2, \ldots, k\}$ and $f^{-1}(h_i) \leqslant \bigvee_{j=1}^{i} g_j$. Then it can be verified that $\{g_i : i = 1, 2, \ldots, k\}$ is a 1*-subshading of X and hence X is countably 1*-compact.

(4) Let $\{g_\lambda : \lambda \in \Lambda\}$ be an open 1*-shading of X. As in (1), we construct an open 1*-shading $\{h_y : y \in Y\}$ of Y. Since Y is 1*-paracompact, there is a precise locally finite 1*-shading open refinement, say $\{a_y : y \in Y\}$ of $\{h_y : y \in Y\}$ [8, Theorem 3.8]. For each $\lambda \in M(y)$ and $y \in Y$, let $g_{y,\lambda} = f^{-1}(a_y) \wedge g_\lambda$. Then $\{g_{y,\lambda} : \lambda \in M(y), y \in Y\}$ is an open 1*-shading of X and is a refinement of $\{g_\lambda : \lambda \in \Lambda\}$. Further we show that the family $\{g_{y,\lambda} : \lambda \in M(y), y \in Y\}$ is locally finite: Let $x_0 \in X$, then $f(x_0) \in Y$, and since $\{a_y : y \in Y\}$ is locally finite there exists an open fuzzy set b such that $b(f(x_0)) = 1$ and $a_y \leqslant 1 - b$ holds for all but at most finitely many $y \in Y$. Then $f^{-1}(b)$ is an open fuzzy set in X such that $f^{-1}(b)(x_0) = b(f(x_0)) = 1$, and $f^{-1}(a_y) \leqslant f^{-1}(1 - b) = 1 - f^{-1}(b)$ holds for all but at most finitely many $y \in Y$. Since $M(y)$ is finite for each $y \in Y$, $f^{-1}(a_y) \wedge g_\lambda \leqslant f^{-1}(a_y) \leqslant 1 - f^{-1}(b)$, holds for all but at most finitely many indices (y, λ) where $y \in Y$ and $\lambda \in M(y)$, and hence $g_{y,\lambda} \leqslant 1 - f^{-1}(b)$ holds for all but at most finitely many (y, λ) and thus $\{g_\lambda : \lambda \in \Lambda\}$ has a locally finite open refinement $\{g_{y,\lambda} : \lambda \in M(y), y \in Y\}$ which is 1*-shading of X. Hence X is 1*-paracompact.

5. Local Compactness

A FTS (X, T) is said to be locally compact in the sense of Wong, if for every fuzzy point p in X, there exists an open fuzzy set g such that $p \in g$ and g is compact (in the sense of Chang). A FTS (X, T) is said to be locally compact in the sense of Christoph iff for each fuzzy point p in X, there is a compact fuzzy set h and an open fuzzy set g such that $p \in g \leqslant h$. This is weaker than the definition by Wong.

Note that the concept of a "fuzzy point" and the definition of $p \in a$ where p is a fuzzy point and a, a fuzzy set, do not reduce to their corresponding definitions, if all the fuzzy sets are restricted to take values only 0 and 1. Also the theory of α-compactness due to Gantner *et al.* [5] is the most successful theory of compactness in fuzzy topological spaces. That is why the local α-compactness due to Gantner *et al.* is more natural than the above two definitions. This definition of local α-compactness involves the idea of the support of a fuzy set. If a is a fuzzy subset of a set X, then by the support of a, denoted by $\operatorname{Supp} a$, we mean the crisp subset of X given by $\operatorname{Supp} a = \{x \in X : a(x) > 0\}$.

The following results regarding the support will be useful.

THEOREM 5.1. *Let X be a set. The following statements hold:*

(1) $a \leqslant \operatorname{Supp} a$, *for any fuzzy set a in X.*

(2) $a \leqslant b$ *implies* $\operatorname{Supp} a \subset \operatorname{Supp} b$, *for any two fuzzy sets a, b in X.*

(3) $\operatorname{Supp} a = a$ *iff a is a crisp subset of X.*

(4) $\operatorname{Supp}(\operatorname{Supp} a) = \operatorname{Supp} a$, *for any fuzzy set a in X.*

(5) $\operatorname{Supp}(\bigvee_{\lambda \in \Lambda} a_\lambda) = \bigcup_{\lambda \in \Lambda} \operatorname{Supp} a_\lambda$, *for any family of fuzzy sets $\{a_\lambda : \lambda \in \Lambda\}$ in X.*

(6) $\operatorname{Supp}(\bigwedge_{i=1}^{n} a_i) = \bigcap_{i=1}^{n} \operatorname{Supp} a_i$, *for any family of fuzzy sets $\{a_i : i = 1, 2, ..., n\}$ in X.*

(7) *If $f : X \to Y$ is a function from a set X into a set Y, and a is a fuzzy set in X, then* $f(\operatorname{Supp} a) = \operatorname{Supp} f(a)$.

(8) *If $f : X \to Y$ is a function from a set X into a set Y and g is a fuzzy set in Y, then* $f^{-1}(\operatorname{Supp} g) = \operatorname{Supp} f^{-1}(g)$.

(9) *For any two fuzzy sets, a, b in X,* $\operatorname{Supp}(a \times b) = \operatorname{Supp} a \times \operatorname{Supp} b$, *where $a \times b$ is a fuzzy set in $X \times Y$ given by $(a \times b)(x, y) = a(x) \wedge b(y)$ for all $(x, y) \in X \times Y$.*

Proof. We prove only (7). Let $y \in f(\operatorname{Supp} a)$. Then there exists $x_0 \in X$ such that $x_0 \in \operatorname{Supp} a$, hence $a(x_0) > 0$, and $f(x_0) = y$. Therefore, $\bigvee \{a(x) : x \in f^{-1}(y)\} > 0$, that is, $\bigvee \{a(x) : f(x) = y\} > 0$ which implies that $f(a)(y) > 0$ and hence $y \in \operatorname{Supp} f(a)$ and these steps can be retraced to show that $\operatorname{Supp} f(a) \subset f(\operatorname{Supp} a)$.

Easy verification of the remaining results is left for the reader.

We refer to the definition of local α-compactness due to Gantner *et al.* [5] as strongly locally α-compact and recall.

DEFINITION 5.2 [5]. Let $0 \leqslant \alpha < 1$ $(0 < \alpha \leqslant 1)$. A FTS (X, T) is said to be strongly locally α- (α^*-) compact if for each point p in X, there is an open neighbourhood n of p such that $n(p) = 1$ and $\operatorname{Supp} \bar{n}$ is α-compact (α^*-compact).

Now we give the following.

DEFINITION 5.3. Let $0 \leqslant \alpha < 1$ $(0 < \alpha \leqslant 1)$. A FTS (X, T) is said to be locally α-compact (locally α^*-compact) iff for each point p in X there is a neighbourhood n of p such that $n(p)=1$ and Supp n is α-compact (α^*-compact).

Note that a strongly locally α-compact FTS (X, T) is locally α-compact. But the converse need not be true as the following example shows.

EXAMPLE 5.4. Let X be an infinite set and let T include μ_ϕ, μ_X, along with the characteristic function of each subset of X that contains a particular point p. Then T is a fuzzy topology on X (in fact, a particular point topology on X). In such a space the closure of any open fuzzy set other than μ_ϕ is μ_X. Note that (X, T) is locally α-compact. For if $z \in X$, then there is a neighbourhood n_z of z given by

$$\begin{aligned} n_z(x) &= 1 \qquad \text{if} \quad x=z \text{ or } x=p \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

such that Supp $n_z=\{z,p\}$ which is α-compact. (X, T) is not strongly locally α-compact. For if $x \in X$ and if n is any open neighbourhood of x, then $\bar{n}=\mu_X$ so that Supp $\bar{n}=X$ which is not α-compact since X is infinite.

The equivalence of the two notions of local α-compactness is given in the following:

THEOREM 5.5. *Let (X, T) be a 1^*-Hausdorff* FTS *such that* $\operatorname{Supp} \bar{a} = \overline{\operatorname{Supp} a}$ *for each $a \in T$. Then (X, T) is strongly locally α-compact if and only if it is locally α-compact.*

Proof. We need only to show that local α-compactness implies strong local α-compactness. Let (X, T) be locally α-compact. Let $x \in X$. Then there is a neighbourhood n of x such that $n(x)=1$ and Supp n is α-compact. Then there exists an open fuzzy set g such that $g \leqslant n$ and $g(x)=n(x)=1$. Now we have $\overline{\operatorname{Supp} g} \subset \overline{\operatorname{Supp} n}$. But Supp n is closed from [5, Theorem 2.8] and so $\overline{\operatorname{Supp} g} \subset \operatorname{Supp} n$. Thus $\overline{\operatorname{Supp} g}$ is a closed crisp subset of Supp n which is α-compact. Then $\overline{\operatorname{Supp} g} = \operatorname{Supp} \bar{g}$ is α-compact from [5, Theorem 2.6]. Hence (X, T) is strongly locally α-compact.

Analogous statement holds for the α^*-case.

THEOREM 5.6. *Let F be a closed crisp subset of a* FTS (X, T). *If X is locally α-compact (locally α^*-compact), then F is locally α-compact (locally α^*-compact) as a crisp subspace of X.*

Proof. Let $x \in F$, then since X is locally α-compact there is a neighbourhood n of x in X such that $n(x)=1$ and Supp n is α-compact. Now

$n \wedge F$ is a neighbourhood of x in F and $(n \wedge F)(x) = 1$. It remains to show that $\operatorname{Supp}(n \wedge F)$ is α-compact. But $\operatorname{Supp}(n \wedge F) = \operatorname{Supp} n \cap \operatorname{Supp} F = (\operatorname{Supp} n) \cap F$. Thus it suffices to show that $(\operatorname{Supp} n) \cap F$ is α-compact. Let $\mathscr{G}$ be an open α-shading of $(\operatorname{Supp} n) \cap F$ in F. Now if $g \in \mathscr{G}$, then there is a $g^* \in T$ such that $g = g^* \wedge F$. Let $\mathscr{G}^* = \{g^* : g^* \wedge F \in \mathscr{G}\}$. Then $\mathscr{G}^* \cup \{X - F\}$ is an open α-shading of $\operatorname{Supp} n$ in X: For let $z \in \operatorname{Supp} n$. If $z \notin F$, then $(X - F)$ is an open set such that $(X - F)(x) = 1 > \alpha$. If $z \in F$, then $z \in (\operatorname{Supp} n) \cap F$ and $\mathscr{G}$ is an open α-shading of $(\operatorname{Supp} n) \cap F$. Therefore, there exists $g = g^* \wedge F$ in $\mathscr{G}$ such that $g(z) > \alpha$. This implies that there exists $g^* \in \mathscr{G}^*$ such that $g^*(z) \geqslant (g^* \wedge F)(z) = g(z) > \alpha$. Hence by α-compactness of $\operatorname{Supp} n$, there is a finite α-subshading of $\operatorname{Supp} n$ from $\mathscr{G}^* \cup \{X - F\}$, say $\{g_1^*, g_2^*, ..., g_m^*, X - F\}$. Then $\{g_1, g_2, ..., g_m\}$ is a finite α-subshading of $(\operatorname{Supp} n) \cap F$ from $\mathscr{G}$, and $(\operatorname{Supp} n) \cap F = \operatorname{Supp}(n \wedge F)$ is α-compact. Hence F is locally α-compact. The proof of the α^*-case is similar.

THEOREM 5.7. *Let $f: X \to Y$ be an F-continuous, F-open surjection. If X is locally α-compact (locally α^*-compact), then Y is also locally α-compact (locally α^*-compact).*

Proof. Let $y \in Y$ and let $f(x) = y$. Since $x \in X$ there exists a neighbourhood n of x such that $n(x) = 1$ and $\operatorname{Supp} n$ is α-compact. Since f is F-open, $f(n)$ is a neighbourhood of y such that $f(n)(y) = \bigvee \{n(x) : f(x) = y\} = 1$ and since f is F-continuous, $f(\operatorname{Supp} n)$ is α-compact. But $f(\operatorname{Supp} n) = \operatorname{Supp} f(n)$ so that $\operatorname{Supp} f(n)$ is α-compact and hence Y is locally α-compact. The proof of the α^*-case is similar.

From Theorems 5.5 and 5.7, we have the following.

COROLLARY 5.8. *Let f be an F-continuous, F-open function from a* FTS *X onto a 1^*-Hausdorff* FTS *Y such that $\operatorname{Supp} \bar{a} = \overline{\operatorname{Supp} a}$ for each open fuzzy set a in Y. If X is strongly locally α-compact (strongly locally α^*-compact), then so is Y.*

The product fuzzy topological spaces were introduced by Wong [13].

THEOREM 5.9. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of nonempty* FTS. *Then the product* FTS *$\prod_{\lambda \in \Lambda} X_\lambda$ is locally α-compact iff each X_λ is locally α-compact and all but finitely many X_λ are α-compact.*

Proof. Let X_λ be locally α-compact for each λ, and X_λ be α-compact for all λ except $\lambda = \lambda_1, \lambda_2, ..., \lambda_k$. Let $x \in \prod_{\lambda \in \Lambda} X_\lambda$. Then $P_\lambda(x) = x_\lambda \in X_\lambda$, where P_λ is the λth projection map, and by local α-compactness of X_λ, there exists a neighbourhood n_λ of x_λ such that $n_\lambda(x_\lambda) = 1$ and $\operatorname{Supp} n_\lambda$ is

α-compact. The fuzzy set $g = \bigwedge_{i=1}^{k} p_{\lambda_i}^{-1}(n_{\lambda_i}) = n_{\lambda_1} \times n_{\lambda_2} \times \cdots \times n_{\lambda_k} \times \pi\{X_\lambda : \lambda \neq \lambda_1, \lambda_2, ..., \lambda_k\}$ is a neighbourhood of x in $\prod_{\lambda \in \Lambda} X_\lambda$, and

$$g(x) = \left[\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(n_{\lambda_i})\right](x) = \bigwedge_{i=1}^{k} [P_{\lambda_i}^{-1}(n_{\lambda_i})](x)$$

$$= \bigwedge_{i=1}^{k} n_{\lambda_i}(P_{\lambda_i}(x)) = \bigwedge_{i=1}^{k} n_{\lambda_i}(x_{\lambda_i}) = 1.$$

Also $\operatorname{Supp} g = \operatorname{Supp}[\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(n_{\lambda_i})] = \bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}[\operatorname{Supp} n_{\lambda_i}]$ which, by the Tychonoff product theorem [5, Theorem 3.4], is α-compact since each $\operatorname{Supp} n_{\lambda_i}$, $i = 1, 2, ..., k$, is α-compact and $X_\lambda, \lambda \neq \lambda_1, \lambda_2, ..., \lambda_k$ are α-compact.

On the other hand, let πX_λ be locally α-compact. Since $P_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \to X_\lambda$ is F-continuous and F-open surjection [12], by Theorem 5.7, X_λ is locally α-compact for each λ. Now let $x \in \prod_{\lambda \in \Lambda} X_\lambda$, then there exists a neighbourhood g of x in $\prod_{\lambda \in \Lambda} X_\lambda$ such that $g(x) = 1$ and $\operatorname{Supp} g$ is α-compact. Then there is a basic open fuzzy set of the form $O_{\lambda_1} \times O_{\lambda_2} \times \cdots \times O_{\lambda_k} \times \Pi\{X_\lambda : \lambda \neq \lambda_1, \lambda_2, ..., \lambda_k\} = \bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(O_{\lambda_i})$ such that $\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(O_{\lambda_i}) \leqslant g$ and $[\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(O_{\lambda_i})](x) = g(x) > 0$. Then $\operatorname{Supp} g \geqslant \operatorname{Supp}[\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(O_{\lambda_i})] = \bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(\operatorname{Supp} O_{\lambda_i})$. Then for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_k$, $P_\lambda(\operatorname{Supp} g) \geqslant P_\lambda[\bigwedge_{i=1}^{k} P_{\lambda_i}^{-1}(\operatorname{Supp} O_{\lambda_i})] = X_\lambda$. Since P_λ is F-continuous, and $\operatorname{Supp} g$ is α-compact, X_λ is α-compact for all λ except possibly $\lambda = \lambda_1, \lambda_2, ..., \lambda_k$. This completes the proof.

6. Star Products

Let X and Y be two FTS. Then the star product fuzzy topology on $X \times Y$ has a base consisting of all fuzzy sets of the form $a * b$, where a is an open fuzzy set in X and b is an open fuzzy set in Y, $a * b$ being defined by the formula,

$$(a * b)(x, y) = a(x) \cdot b(y) \qquad \text{for all} \quad (x, y) \in X \times Y.$$

When $X \times Y$ is equipped with this fuzzy topology it is denoted by $X * Y$ and is called the star product of the FTS X and Y. Such products were studied in [5 and 6].

THEOREM 6.1. *If X and Y are locally* 0*-compact* FTS *then $X * Y$ is locally* 0*-compact.*

Proof. Let $(x, y) \in X \times Y$. Then by hypothesis, there is a neighbourhood g of x in X such that $g(x) = 1$ and $\operatorname{Supp} g$ is 0-compact, and there is a neighbourhood h of y in Y such that $h(y) = 1$ and $\operatorname{Supp} h$ is 0-compact. Then $(g * h)(x, y) = g(x) \cdot h(y) = 1$. Now since $\operatorname{Supp} g$ and $\operatorname{Supp} h$ are 0-

compact we have (Supp g) $*$ (Supp h) is 0-compact from [5, Theorem 4.1]. But $\mathrm{Supp}(g * h) = (\mathrm{Supp}\, g) * (\mathrm{Supp}\, h)$, and therefore, $\mathrm{Supp}(g * h)$ is 0-compact. Thus there is a neighbourhood $(g * h)$ of (x, y) in $X * Y$ such that $(g * h)(x, y) = 1$ and $\mathrm{Supp}(g * h)$ is 0-compact. Hence $X * Y$ is locally 0-compact.

The following theorem can be easily proved.

THEOREM 6.2. *If X and Y are any two α-Hausdorff (α^*-Hausdorff) fuzzy topological spaces, then $X * Y$ is α-Hausdorff (α^*-Hausdorff).*

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